Computing Competitive Equilibrium for Chores: Linear Convergence and Lightweight Iteration

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Outline

Background

DCA for Computing CE for Chores

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Economic Background: Allocation of Chores

There are many settings when we need to (fairly) allocate shared chores to users.

- ◇ Job shifts among workers.
- ◇ Papers among reviewers.

Figure 1: Teaching load among faculty

Fair Chores Allocation ⇒ CE for Chores

Definition (Competitive Equilibrium for Chores)

A price $p \in \mathbb{R}_+^m$ and an allocation $x \in \mathbb{R}_+^{n\times m}$ satisfy competitive equilibrium (CE) if and only if

(E1). $p^\top x_i = B_i$ for all $i \in [n]$;

(E2). $d_i^\top x_i \le d_i^\top y_i$ for all $y_i \in \mathbb{R}_+^m$ such that $p^\top y_i \ge p^\top x_i$, for all $i \in [n]$;

(E3). $\sum_{i\in[n]} x_{ij} = 1$ for all $j \in [m]$.

Remark: Condition (E2) says that agent i minimizes his disutility under his expected amount. This is equivalent to that agent i only chooses chores from the set $\arg\max_j \{\frac{p_j}{d_{ij}}\}$.

Remark: CE for chores ⇔ Agent optimality + Market clearance.

From Exact CE to ϵ-CE

Definition (Approximate CE for Chores)

A price $p \in \mathbb{R}_+^m$ and an allocation $x \in \mathbb{R}_+^{n \times m}$ satisfy ϵ-CE if and only if

(E1). $(1-\epsilon)B_i \le p^\top x_i \le \frac{1}{1-\epsilon} B_i$ for all $i \in [n]$;

(E2). $(1-\epsilon)d_i^\top x_i \le d_i^\top y_i$ for all $y_i \in \mathbb{R}_+^m$ such that $p^\top y_i \ge p^\top x_i$, for all $i \in [n]$;

(E3). $1 - \epsilon \le \sum_{i \in [n]} x_{ij} \le \frac{1}{1-\epsilon}$ for all $j \in [m]$.

Remark: Even when ϵ is very small, an ϵ-CE can still be far from an exact CE [Chaudhury et al., 2024]!

CE for Chores ⇔ KKT of Problem (CE)

$$\max_{\beta \in \mathbb{R}^n_+, p \in \mathbb{R}^m_+} \quad \sum_{j \in [m]} p_j - \sum_{i \in [n]} B_i \log(\beta_i) \qquad \text{subject to} \quad p_j \le \beta_i d_{ij} \quad \forall\ i \in [n], j \in [m]. \tag{CE}$$

Algorithm	Result	Remark
Polynomial-time algorithm [Garg and Végh, 2019]	PPAD-hardness in the exchange model	Contrast with goods setting
First-order methods [Bogomolnaia et al., 2017]	Converge to the boundary	Undesirable allocation
Combinatorial algorithm [Chaudhury and Mehlhorn, 2018]	Time complexity $\tilde{\mathcal{O}}(nm/\epsilon^2)$	Numerical issue
Exterior-point method [Boodaghians et al., 2022]	$\tilde{\mathcal{O}}(n^3/\epsilon^2)$ QPs	ϵ-CE
Greedy Frank-Wolfe [Chaudhury et al., 2024]	$\tilde{\mathcal{O}}(n/\epsilon^2)$ LPs	ϵ-CE

Table 1: Prior Arts in Computing CE for Chores

Motivations of Our Work

Questions:

- How can we move closer to the **exact** CE for chores?
- How can we **avoid solving subproblems** when computing CE for chores?

Our Answers - Problem Reformulation

Step 1: Add redundant constraint [Chaudhury et al., 2024]:

$$\begin{aligned} \min_{p \in \mathbb{R}_+^m} \quad & \sum_{i \in [n]} B_i \log\left(\max_{j \in [m]} \left\{\frac{p_j}{d_{ij}}\right\}\right) \\ \text{subject to} \quad & \sum_{j \in [m]} p_j = \sum_{i \in [n]} B_i. \end{aligned} \tag{CE-1}$$

Step 2: Remove constraint:

$$\min_{p \in \mathbb{R}_+^m} \quad \sum_{i \in [n]} B_i \log\left(\max_{j \in [m]} \left\{\frac{p_j}{d_{ij}}\right\}\right) - \sum_{i \in [n]} B_i \log\left(\sum_{j \in [m]} p_j\right). \tag{CE-2}$$

Step 3: Variable replacement:

$$\min_{\mu \in \mathbb{R}^m} \quad \sum_{i \in [n]} B_i \max_{j \in [m]} \left\{\mu_j - \log(d_{ij})\right\} - \sum_{i \in [n]} B_i \log\left(\sum_{j \in [m]} e^{\mu_j}\right). \tag{DC}$$

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Approximating g: DCA Difference-of-Convex Algorithm

The problem (DC)

$$\min_{\mu \in \mathbb{R}^m} \quad f(\mu) - g(\mu)$$

where

$$f(\mu) \coloneqq \sum_{i \in [n]} B_i \max_{j \in [m]} \left\{ \mu_j - \log(d_{ij}) \right\} + \frac{\eta}{2} \|\mu\|^2,$$

$$g(\mu) \coloneqq \sum_{i \in [n]} B_i \log \left(\sum_{j \in [m]} e^{\mu_j} \right) + \frac{\eta}{2} \|\mu\|^2.$$

In each iteration, we solve

$$\min_{\mu \in \mathbb{R}^m} \quad f(\mu) - \nabla g(\mu^k)^\top (\mu - \mu^k)$$

Main Theorem for DCA

Theorem (Local Linear Convergence)

Let $\{\mu^k\}_{k\geq 0}$ be the sequence generated by the DCA with regularization coefficient η. Then $\{\mu^k\}_{k\geq 0}$ converge R-linearly to a stationary point.

- **Sufficient descent**
  $F(\mu^{k+1}) - F(\mu^k) \leq -\frac{\eta}{2}\|\mu^k - \mu^{k+1}\|^2$
- **Relative error**
  $\exists\ u^{k+1} \in \partial F(\mu^{k+1}), \|u^{k+1}\| \leq \left(\eta + \sum_{i\in[n]} B_i\right)\|\mu^{k+1} - \mu^k\|$
- **Lower boundedness**
  $\mathbf{1}^\top \mu^k \equiv \mathbf{1}^\top \mu^0$
- **Local regularity**

Further Remarks - Local Regularity

Theorem (Local Error Bound)

Let $\mathcal{U}^*$ be the set of stationary points of (DC) *and $\bar{\mu} \in \mathcal{U}^*$. There exist constants $\delta > 0$ and $\tau > 0$ such that for all μ with $\|\mu - \bar{\mu}\| \leq \delta$,*

$$\operatorname{dist}(\mu, \mathcal{U}^*) \leq \tau \operatorname{dist}(0, \partial F(\mu)).$$

Theorem (KL exponent of $1/2$)

Problem (DC) *satisfies the KL property with an exponent of $1/2$, i.e., for every $\bar{\mu} \in \mathbb{R}^m$, there exist constants $\epsilon, \eta, \nu > 0$ such that*

$$F(\mu) - F(\bar{\mu}) \leq \eta \cdot \operatorname{dist}(0, \partial F(\mu))^2,$$

whenever $\|\mu - \bar{\mu}\| \leq \epsilon$ and $F(\bar{\mu}) < F(\mu) < F(\bar{\mu}) + \nu$.

Further Remarks - Computation

Subproblem of DCA

$$\begin{aligned} \min_{\lambda \in \mathbb{R}_+^{m \times n}} \quad & \frac{1}{2} \sum_{j \in [m]} \| \sum_{i \in [n]} \lambda_{ij} - \frac{1}{\eta} \nabla_j g(\mu^k) \|^2 + \log(D) \bullet \lambda \\ \text{subject to} \quad & \sum_{j \in [m]} \lambda_{ij} = \frac{1}{\eta} B_i, \ \forall i \in [n]. \end{aligned}$$

- ◇ Projected gradient descent ⇒ lightweight projection.
- ◇ Mirror descent ⇒ analytical update.

Outline

Approximating f: SGR
Smoothing Gradient Descent with Rounding

Approximating $\max$ via entropy term

$$\min_{\mu \in \mathbb{R}^m} \quad \sum_{i \in [n]} B_i \max_{\lambda_i \in \Delta_m} \Big\{ \sum_{j \in [m]} \lambda_{ij} \left(\mu_j - \log(d_{ij}) \right) - \delta \lambda_{ij} \log(\lambda_{ij}) \Big\} - \sum_{i \in [n]} B_i \log \Bigg(\sum_{j \in [m]} e^{\mu_j} \Bigg).$$

$$\min_{\mu \in \mathbb{R}^m} \quad F_\delta(\mu) \coloneqq \delta \sum_{i \in [n]} B_i \log \Bigg(\sum_{j \in [m]} e^{\frac{\mu_j - \log(d_{ij})}{\delta}} \Bigg) - \sum_{i \in [n]} B_i \log \Bigg(\sum_{j \in [m]} e^{\mu_j} \Bigg).$$

Fact (Properties of F_δ)

The following properties hold:
(i) $F \leq F_\delta \leq F + \delta \log(m) \sum_{i \in [n]} B_i$,
(ii) $\nabla F_\delta(\mu)$ *can be evaluated in* $\mathcal{O}(mn)$ *time,*
(iii) ∇F_δ *is* $\sum_{i \in [n]} B_i (1/\delta + 1)$ *Lipschitz continuous.*

Next Step: Guarantee for Gradient Descent?

Lemma (ϵ-CE and $\nabla_j F_\delta(\mu)$)

Let $q_j(\mu) \coloneqq \sum_{i\in[n]} B_i e^{\mu_j} / \sum_{j\in[m]} e^{\mu_j}$

$|\nabla_j F_\delta(\mu)/q_j(\mu)| \le \epsilon$ *for all* $j \in [m]$, *and* $\epsilon \ge (1.3 + \log(m-1))\delta$.

Then (p, x) *with* $p_j = q_j(\mu)$ *and* $x_{ij} = v_{ij}/p_j$ *is an* ϵ*-CE.*

Remark: We need rounding to derive the nonasymptotic results!

Fact (Basis for Rounding)

Suppose that $\delta \le 1/(2 + \log(m-1))$ *and* $q_{j_0}(\mu) < e^{\underline{\mu_\delta}}$, *where*

$$\underline{\mu_\delta} \coloneqq \log\left(\frac{\sum_{i\in[n]} B_i}{2m}\right) - \frac{1+\delta}{1-\delta}\log\left(\frac{\max_{ij}\{d_{ij}\}}{\min_{ij}\{d_{ij}\}}\right) - \delta\log(4m).$$

Then $\nabla_{j_0} F_\delta(\mu) < 0$.

From Rounding to Non-Asymptotic Results

Lemma (Rounding)

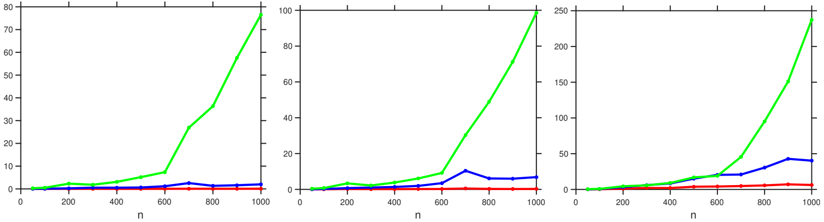
Rounding algorithm stops in m steps and outputs vector $\mu \in \mathbb{R}^m$
(i) $\mathbf{1}^\top \mu = \mathbf{1}^\top \mu^0$;
(ii) $F_\delta(\mu) \leq F_\delta(\mu^0)$;
(iii) $q_j(\mu) \geq e^a$ *for all* $j \in [m]$.

Theorem (Non-Asymptotic Results)

The SGR finds an ϵ-CE in at most $\tilde{\mathcal{O}}(\frac{m^2}{\epsilon^3})$ iterations, and the total complexity is at most $\tilde{\mathcal{O}}(\frac{m^3(n+m)}{\epsilon^3})$.

Outline

Numerical Results



(a) Log-Uniform (b) Exponential (c) Integer

Figure 2: CPU Time Comparison under Different Generative Models, $\epsilon = 0.01$.

Outline

Closing Remarks

This paper:

- ✓ DCA: Linear convergence to exact CE.
- ✓ SGR: Subproblem-free algorithm for computing ϵ-CE.

Future directions:

- ? CE for chores: extension to different utility function in computing CE for chores.
- ✓ CE for goods: working paper by our group.
- ? CE for X: efficient algorithm for more complex market [Jalota et al., 2023].

Thanks!

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